CHARACTERISTICS OF FUNCTIONS AND INVERSE FUNCTIONS

DEFINITION: Suppose $F : X \to Y$.

- F is called **one-to-one** or **injective** iff $x_1 \neq x_2 \implies F(x_1) \neq F(x_2)$. In this case, we write $F : X \rightarrowtail Y$

  Equivalently, F is injective iff $F(x_1) = F(x_2) \implies x_1 = x_2$.

- F is called **onto** or **surjective** iff $F(X) = Y$. In this case, we write $F : X \twoheadrightarrow Y$

- F is called **bijective** iff F is both injective and surjective. In this case, we write $F : X \rightarrowtail\!\!\!\!\rightarrow Y$

EXAMPLE: Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$.

- Give an example of a function $F : X \to Y$ which is neither injective nor surjective.

- Give an example of a function $F : X \to Y$ which is surjective.

- Why can't we find an example of $F : X \to Y$ which is injective?

  NOTE: You may want to research the 'pigeon hole principle.'

EXAMPLE: Let $F : X \to Y$. Prove the following:

- F is injective iff $A = F^{-1}(F(A))$ for all $A \subseteq X$.

- F is surjective iff $C = F(F^{-1}(C))$ for all $C \subseteq Y$.

EXAMPLE: Under what circumstances is:

- $F(A \cap B) = F(A) \cap F(B)$ for all $A, B \subseteq X$?

- $F(X \backslash A) = Y \backslash F(A)$ for all $A \subseteq X$?

EXAMPLE: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$.

Prove F is open iff F is closed.

EXAMPLE: If $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$, define G as follows: for all $y \in Y$, $G(y) = x$ where $x \in X$ with $F(x) = y$.

Prove $G : Y \to X$ is a function.

DEFINITION: If $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$, the **inverse of** F, denoted F^{-1} is defined as: $F^{-1}(y) = x \iff F(x) = y$.

EXAMPLE: Suppose $F : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y$. Prove the following:

- $F^{-1} : Y \to X$ is a bijection.

- $\left(F^{-1}\right)^{-1} = F$.

EXAMPLE: If $F : X \to Y$ and $G : Y \to Z$, define H as follows: for $x \in X$, $H(x) = G(F(x))$.

Prove H is a function: $H : X \to Z$.

DEFINITION: Given $F : X \to Y$ and $G : Y \to Z$, we define $G \circ F : X \to Z$ as $(G \circ F)(x) = G(F(x))$.

We may sketch out the relationship between F, G and $G \circ F$ by sketching a so-called 'commutative' diagram:

EXAMPLE: Let $F : \mathbb{R} \to [0, \infty)$ be given by $F(x) = x^2$ and let $G : [0, \infty) \to [0, \infty)$ be given by $G(x) = \sqrt{x}$.

- Find $G \circ F$.

- Find $F \circ G$.

EXAMPLE: Suppose $F : X \to Y$ and $G : Y \to Z$.

- If F and G are injective, prove that $G \circ F$ is injective.

- If F and G are surjective, prove that $G \circ F$ is surjective.

EXAMPLE: Suppose $F : X \to Y$ and $G : Y \to Z$.

- If $G \circ F$ is injective, what can be said about F? G?

- If $G \circ F$ is surjective, what can be said about F? G?

DEFINITION: Given a set X, we define the **identity** function $1_X : X \to X$ to be $1_X(x) = x$ for all $x \in X$.

EXAMPLE: Suppose $F : X \to Y$. Prove:

- $F \circ 1_X = F$

- If F^{-1} exists, then $F^{-1} \circ F = 1_X$.

EXAMPLE: Suppose $F : X \to Y$. Prove:

- F is injective iff there exists a surjective function G such that $G \circ F = 1_X$.

  NOTE: G is called a 'left-inverse' of F.

- F is surjective iff there exists an injective function G such that $F \circ G = 1_Y$.

  NOTE: G is called a 'right-inverse' of F. Also, you may want to research the Axiom of Choice ...

- F is bijective iff there exists a **unique** function G such that $G \circ F = 1_X$ and $F \circ G = 1_Y$.

  NOTE: Here, G is called a 'two-sided inverse' of F. Because of uniqueness, $G = F^{-1}$ as defined earlier.

EXAMPLE: $F : \mathbb{R} \to [0, \infty)$ given by $F(x) = x^2$ is surjective.

- Show $G : [0, \infty) \to [0, \infty)$ given by $G(x) = \sqrt{x}$ is a right inverse of F.

- Show $H : [0, \infty) \to (-\infty, 0]$ given by $H(x) = -\sqrt{x}$ is also a right inverse of F. .

EXAMPLE: Suppose $F : X \rightarrowtail\!\!\!\!\rightarrow Y$ and $G : Y \rightarrowtail\!\!\!\!\rightarrow Z$.

- Prove $G \circ F : X \rightarrowtail\!\!\!\!\rightarrow Z$.

- Prove $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

CANTOR-SCHROEDER-BERNSTEIN THEOREM:

If X and Y are sets and there exists $F : X \rightarrowtail Y$ and $G : Y \rightarrowtail X$, then there exists $H : X \rightarrowtail\!\!\!\!\rightarrow Y$